Networking process and fractal dimensionality in percolation

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The growth mechanism is studied for percolation networks on a square lattice by means of a Monte Carlo technique. The mechanism near the threshold $p=p_c$ is accompanied with the extensive linking of clusters during the random-increment process of occupied sites on the lattice. Local dimensionality is defined from the magnification rates of linking clusters. The linking mechanism characterizes the process for fractal growth in percolation. [S1063-651X(98)00607-2]

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Since the concept of ''fractal'' was established, geometrical features in nature have been described with it [1]. In the study of fractals, computer simulations have presented a number of realistic models of structures [2]. Percolation clusters simulated with the Monte Carlo (MC) technique have been known to be one of the typical models of the structure with fractal geometry [3]. The dimensionalities on two-dimensional (2D) lattices have been $D=\frac{91}{48}=1.896$ for clusters at the threshold $p=p_c$ and $D=1.56$ for clusters in the noncritical region $p<p_c$ [3].

In order to analyze the detailed structure of percolation clusters, Stanley and Coniglio proposed the ''link-node-blob'' model [4,5]: links are defined as one-dimensional chains, nodes as their crossing points, and blobs as connection chains with two or more junctions between two points. Bonds in the clusters have been classified into three parts: ''red'' is a singly joined backbone bond, ''blue'' a multiply joined one, and ''yellow'' a dangling end [6]. This model has been widely used to study fine structure of the clusters.

The purpose of this paper is to present an interpretation for the origin of the above dimensionalities. We examine the growth process of the clusters on a square (SQ) lattice by means of the MC simulation. The interlinking site between two clusters, which means the ''red'' bond, plays an essential role in understanding the formation mechanism of fractal networks with the dimensionalities.

First an array of vacant sites was constructed on the lattice points of the SQ lattice in 2D. Their total number N of the sites is $N=L\times L$, where L is the spanning size which is equal to the number of sites on a side of the lattice. On the array of the vacant sites, occupied sites were introduced in the system at random. The number N_p of the occupied sites was defined as $N_p=N\times p$, where p is the occupation probability of the sites. From the occupied sites determined at random, the structure of the clusters was analyzed. Here the cluster was identified as a group of the occupied sites connected by nearest-neighbor bonds. This process corresponds to ''site percolation.''

Here let us examine the growth mechanism of percolation clusters on the SQ lattice. First we obtained size variations of the largest cluster constructed in the system. The size of the cluster is defined as the number of occupied sites belonging to the cluster. Figure 1 shows the variations as a function of occupation probability p. The size for a specified p was averaged for different samples with various random-number seeds. Abscissa is for probability p and ordinate for size $S_{\max}$ of the largest cluster. Here the size $S_{\max}$ was normalized by $N=L^2$. Line a in Fig. 1 is for $L=50$, b for $L=100$, and c for $L=400$. Line a is almost constant with $S_{\max}=0$ in the initial region $p<0.4$. This shows that the cluster does not grow in this region, although the number of occupied sites has been increased in proportion to p. The line begins to increase abruptly at about $p=0.5$ and continues to increase until $p=0.7$, indicating the drastic growth of the cluster. The line shows linear relation for the final region $p>0.7$. Lines for the other sizes L vary in the same manner as line a. The onset probability of the drastic increment increases with increasing L.

We examine the growth process of the cluster in more detail. Figure 1 shows the variations of average size $S_{\max}$ for different samples. In order to know details of the growth process of the cluster, we examined the variations of the size of the cluster without averaging. First we determined a random-number seed for a sample and increased occupied sites at random one by one in the sample: the sites were determined by random-number series with the seed. Figure 2 shows the result for a sample with a random-number seed. The system size was chosen to be $L=100$. Line for the sample varies stepwise in this figure. Smooth curve in it

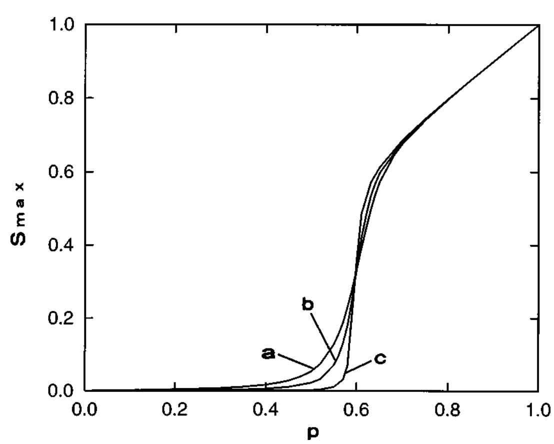

FIG. 1. Variations of average size of the largest clusters constructed on the SQ lattice. The ordinate is the size $S_{\max}$ which is normalized by the total number of sites $N=L^2$ in the system and abscissa is the occupation probability p. Line a is for the system with $L=50$, b with $L=100$, and c with $L=400$.

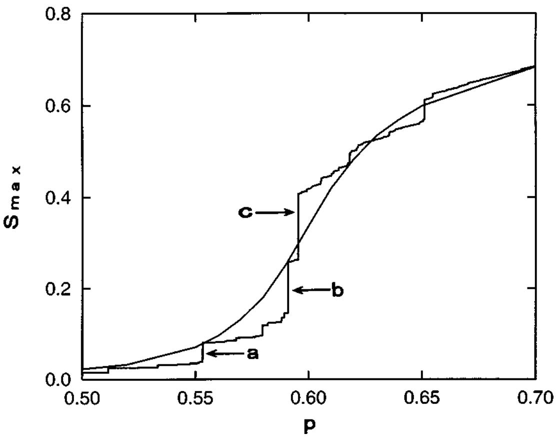

FIG. 2. Variation of nonaverage size of the largest cluster during the random-increment process of occupied sites on the SQ lattice with $L=100$. The size varies stepwise. Smooth curve shows the variation of the average size which is shown as line b in Fig. 1. Arrow a shows the jumping at $p=0.5532$, b at $p=0.5910$, and c at $p=0.5955$.

shows the average size which is shown as line b in Fig. 1. The line in Fig. 2 varies along by the average curve. It is almost constant for $p<0.55$. It begins to increase at about $p=0.55$ and continues to increase stepwise until $p=0.65$. After this, the line is smooth for $p>0.65$.

Why does the step increment appear in Fig. 2? Feder visually observed the formation process of percolation cluster on a 2D lattice: the cluster grew with taking in other clusters as a part of it [7]. Yoshida *et al.* measured sizes of clusters in sintered mixture of Nb and Al_2O_3 with image processing in the laboratory [8]. They observed a percolation behavior of Nb clusters and found their distribution became discrete after the transition of percolation, indicating coupling of the clusters [8]. The coupling behavior has been observed not only in the laboratory but also in computer simulations of the MC method [9]: when a vacant site which is located between clusters is changed to an occupied site, then the clusters are joined by the site.

Let us examine the origin of the step increment appeared in Fig. 2. Arrow a in it shows the jumping when an occupied site is introduced in the sample at $p=0.5532$. Figure 3(a) shows the shape of two clusters at $p=0.5531$ for the sample. Cross marks correspond to the occupied sites belonging to the clusters in the system. The lower cluster is the largest cluster at $p=0.5531$ and the upper one is the cluster near the largest cluster. The lower cluster does not connect with the upper cluster at $p=0.5531$. The small dot in Fig. 3(a) shows the occupied site introduced at $p=0.5532$. The site links the lower cluster to the upper one.

Arrow b in Fig. 2 shows the jumping at $p=0.5910$. This jumping is caused by the linking between the upper and the lower clusters in Fig. 3(b) through the ''red'' site introduced at $p=0.5910$. The extended cluster at $p=0.5910$ spans the system both horizontally and vertically, i.e., from the top to bottom sides and from the left to right sides of the system.

Arrow c in Fig. 2 shows the jumping at $p=0.5955$. This is caused by the linking of two clusters at $p=0.5954$. The two clusters in Fig. 3(c) are linked by the occupied site introduced at $p=0.5955$. The enlarged cluster in Fig. 3(c) covers almost the whole range of the system. Therefore the linking mechanism gives the jumping behavior in the size variation of the largest clusters in Fig. 2.

Here we examine how often the linking of clusters occurs during the random-increment process of occupied sites in the system. There are three situations for occupied sites scattered in the system.

(1) The isolated site which is out of contact with clusters around it. This site contributes to the increment of the total number of clusters in the system.

(2) The contact site with a single cluster in the system. This site is included in the cluster as a part of it and enlarge the cluster size by 1.

(3) The interlinking site between plural clusters. This site joins the clusters in the system and gives the decrement of the total number of clusters in the system.

We can know the situation by the following formula:

$$g(p)=n(p)-n(p-1/L^2), \tag{1}$$

where $n(p)$ is the total number of clusters in the system at p. This formula shows the change of the total number of clusters when an occupied site is introduced in the system at p. If the site is in the situation (1), the formula gives $g(p)=1$. In the case of situation (2), $g(p)=0$ is shown. If it is in the case of (3), it gives $g(p)\leqslant -1$.

We simulated the random-scattering process of occupied sites on the SQ lattice with $L=100$ and identified the situations of the sites. The sites were increased one by one from $p=0$ to $p=1$. Frequency for each situation, which was averaged for 10 samples with different seeds, is shown in Fig. 4. Frequency a is for the situation (1), b for (2), and c for (3). A running average was taken for continuous 10 sites increased in the system. Frequency a is $f=1$ at $p=0$. It decreases with increasing p monotonously.

We find the frequency b increases with p in the initial region $p<0.2$. It begins to saturate at around $p=0.2$ and is almost constant until about $p=0.5$. Afterwards, it begins to increase again and saturates for $p>0.8$. The variation of frequency c shows a bell-type shape: it increases with p until $p=0.5$ and after then it begins to decrease with p. The probability $p=0.5$ giving the maximum value for c corresponds to the onset probability of increment for b.

Figures 1 and 2 have shown that the size of the largest clusters has drastically increased in the probability region between $p=0.55$ and $p=0.65$. This region is on the threshold $p=p_c=0.5927460$ [10] of this lattice for an infinite system size. The effective threshold for a finite system size $L=100$ is $p_c(L=100)=0.58534$ [11] for the SQ lattice, which is also in this region $0.55\leqslant p\leqslant 0.65$. The probability $p=0.5$, at which the frequency c has the maximum value as shown in Fig. 4, does not always coincide with the values of the thresholds for this lattice.

The variation of the total number of clusters constructed in the system has been examined as a function of p. The number had the maximum value at $p=0.3$ which is fairly lower than the threshold $p=p_c$ of this lattice. This can be explained from the variations of the frequency in Fig. 4: the frequency a intersects the frequency c at about $p=0.3$. This probability gives the maximum number of clusters in the system.

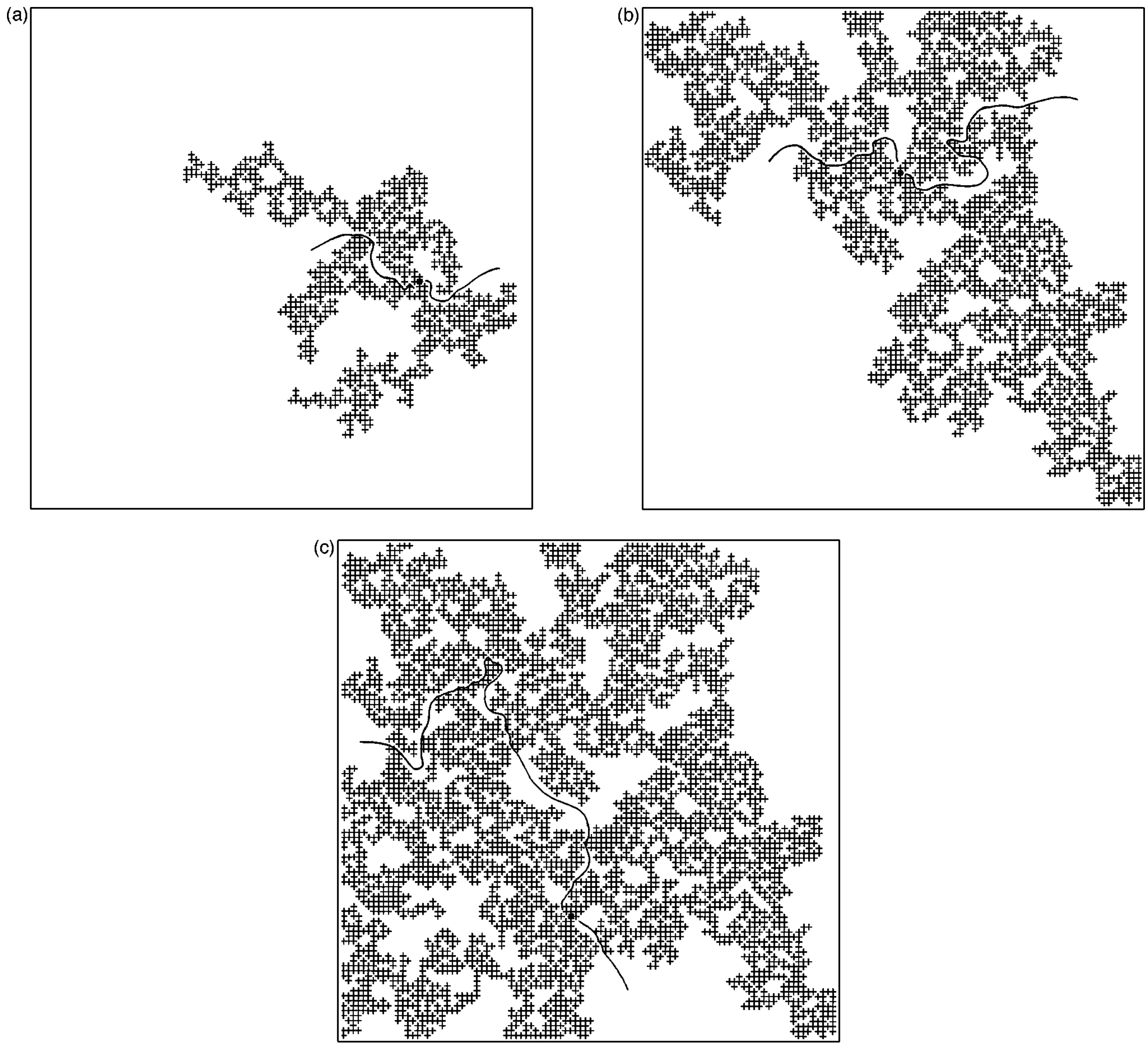

FIG. 3. (a) Shape of two clusters at $p=0.5531$ during the random-increment process of occupied sites on the SQ lattice with $L=100$. The sites belonging to the clusters are shown as crosses. The small filled square shows an occupied site introduced in the system at $p=0.5532$. The site links the lower cluster to the upper one. (b) Shape of two clusters at $p=0.5909$ in the process. The filled square shows an occupied site determined at $p=0.5910$. The site links the lower cluster to the upper one. (c) Shape of two clusters at $p=0.5954$ in the process. The filled square shows an occupied site at $p=0.5955$. The site links the two clusters in the system.

The step increment of the size of the largest cluster has appeared in Fig. 2. Here we examined the variation of the step width as a function of p. Figure 5 shows the variation of the vertical width ΔS averaged for 100 different samples with $L=100$: the width was obtained when the largest cluster was formed by the linking of clusters. It shows a bell-type shape. The arrow in this figure shows the position of the threshold $p=p_c$ of this lattice. The largest width is found to be on the threshold.

Here we examine the origin of the fractal dimensionalities $D=1.896$ for $p=p_c$ and $D=1.56$ for $p<p_c$ in terms of the linking mechanism of clusters. First we define mass ratios λ_k of clusters as follows:

$$\lambda_k=S_1(p)/S_k(p-1/L^2), \quad (k=i,j), \tag{2}$$

where $S_1(p)$ is the size of the largest cluster in the system at p, $S_i(p-1/L^2)$ and $S_j(p-1/L^2)$ are sizes of two clusters which are joined by an occupied site at p. This shows the magnification rates of mass for linking clusters i and j. This is considered as a measure of mass change when the largest cluster is produced by the linking of two clusters.

For the change of spatial size of clusters, we define the radius ratios μ_k as follows:

$$\mu_k=r_1(p)/r_k(p-1/L^2), \quad (k=i,j), \tag{3}$$

where $r_1(p)$ is the radius of the largest cluster at p, and $r_i(p-1/L^2)$ and $r_j(p-1/L^2)$ are radii of two clusters just before linking. The radius of a cluster with size s is obtained from [3,12]

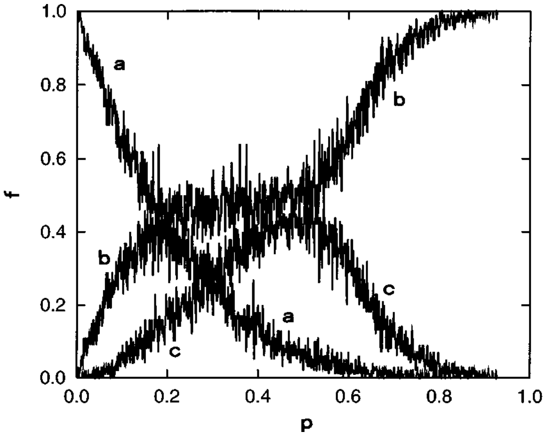

FIG. 4. Variations of frequency of each situation for occupied sites scattered at random in the system. These are drawn as a function of p. Variation a is for the situation (1), b for (2), and c for (3).

$$\mathbf{r}^2=\sum_i |\mathbf{r}_i-\mathbf{r}_0|^2/s=\sum_i \sum_j |\mathbf{r}_i-\mathbf{r}_j|^2/2s^2, \tag{4}$$

where $\mathbf{r}_i$ and $\mathbf{r}_j$ are positions of i and j sites belonging to the cluster, and $\mathbf{r}_0$ is the central position of the cluster given by

$$\mathbf{r}_0=\sum_i \mathbf{r}_i/s. \tag{5}$$

The ratios μ_k show the magnification rates of radii for linking clusters i and j. This is a measure of radius change when the largest cluster is formed by the linking of two clusters.

By use of these ratios λ_k and μ_k, we define the local dimensionalities D_k as follows:

$$D_k=\ln(\lambda_k)/\ln(\mu_k) \quad (k=i,j). \tag{6}$$

When two clusters i and j are linked, the two values D_i and D_j are obtained with this formula.

We simulated a random-increment process of occupied sites from $p=0.5500$ to $p=0.6500$ for the SQ lattice with $L=100$. This range has given the extensive linking of clusters as shown in Fig. 2. We obtained the values D_k when the

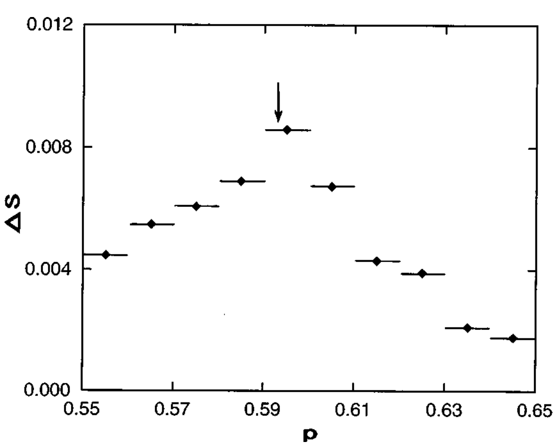

FIG. 5. Variation of step width ΔS in size change of the largest clusters as a function of p. The width ΔS is normalized by $N=L^2$. The arrow shows the position of the threshold $p=p_c$ for the SQ lattice.

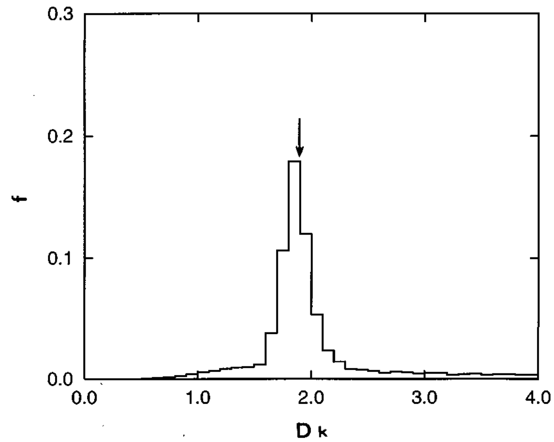

FIG. 6. Distribution of local dimensionalities D_k of clusters defined as formula (6). This is obtained from the random-increment process of occupied sites on the SQ lattice from $p=0.5500$ to $p=0.6500$. The arrow shows the position of the fractal dimensionality $D=\frac{91}{48}=1.896$ of percolation clusters at $p=p_c$ on 2D lattices.

largest clusters were formed by the linking of two clusters in the system. The values were evaluated for the clusters with size $S_k(p-1/L^2)>1$ which gave finite radius $r_k(p-1/L^2)>0$ for the clusters just before linking.

Distribution of the local dimensionalities D_k is shown in Fig. 6 as a histogram. It was obtained from 100 different samples for the system. The histogram is drawn for the range between $D_k=0$ and $D_k=4$ with mesh $\Delta D_k=0.1$. The shape of it is bell type: the highest peak is for the range between $D_k=1.8$ and $D_k=1.9$. The arrow in this figure shows the position of fractal dimensionality $D=\frac{91}{48}=1.896$ of percolation clusters on 2D lattices. The peak position of the histogram coincides with the position of the arrow. This shows that the linking mechanism elucidates the origin of the fractal dimensionality $D=1.896$ for percolation clusters at $p=p_c$.

The dimensionality of clusters has been known to be $D=1.56$ for the noncritical region $p<p_c$ [3]. Here let us examine whether or not the dimensionality $D=1.56$ can be explained by the linking mechanism of clusters. We simulated the random-increment process of occupied sites from $p=0.4000$ to $p=0.5000$ for the SQ lattice with $L=100$. This probability range is near the probability $p=0.5$ at which the maximum value for the frequency c has appeared in Fig. 4. The distribution of D_k is shown as a histogram in Fig. 7. The highest peak is for the range between $D_k=1.5$ and $D_k=1.6$. The arrow shows the position of the dimensionality $D=1.56$ of clusters for $p<p_c$. The arrow is on the highest peak of the histogram. This indicates that the linking mechanism explains the origin of the fractal dimensionality $D=1.56$ for $p<p_c$.

In the last part of this paper, we present some discussions. Figure 6 has shown that the distribution of the local dimensionalities D_k has the highest peak on the theoretically predicted value $D=\frac{91}{48}=1.896$ for percolation clusters at the threshold $p=p_c$. The distribution has been examined for the SQ lattice with $L=100$. Here we find that the system size does not affect the shape of the distribution. We simulated the random-increment process of occupied sites on the SQ lattice with $L=200$: the sites were increased at random one

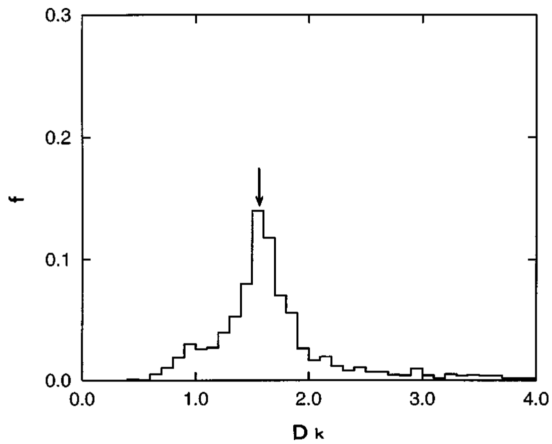

FIG. 7. Distribution of local dimensionalities D_k of clusters defined as formula (6). This is obtained from the random-increment process of occupied sites on the SQ lattice from $p=0.4000$ to $p=0.5000$. The arrow shows the position of the fractal dimensionality $D=1.56$ of clusters for $p<p_c$ on 2D lattices.

by one from $p=0.5500$ to $p=0.6500$ on the lattice. The distribution showed a bell-type shape. The highest peak for $L=200$ was located on the value $D=1.896$.

In Fig. 6, we examined the distribution of the local dimensionalities D_k when the largest cluster was formed by linking of two clusters in the system. There are the other possibility for clusters connected, i.e., linking of three clusters $(L3)$ and of four clusters $(L4)$. We examined how often the linking of clusters occurred with the mechanism of $L3$ and of $L4$. Figure 6 showed the distribution of the dimensionalities D_k when two clusters were linked. It was obtained from 100 different samples for the probability range between $p=0.5500$ and $p=0.6500$. The linking event for this figure was 7641 times. On the other hand, the total number of linking with $L3$ and with $L4$ was found to be only three times. This shows the formation of the largest cluster is rare from the linking of three clusters and of four clusters.

We have clarified that the linking mechanism of clusters gives an interpretation for the origin of the fractal dimensionalities of percolation networks on 2D lattices. This is valid for the other dimensional lattices. Here the distributions of D_k were examined for a simple cubic lattice with $L=30$. The highest peak for $p=p_c$ was found to be located near the value $D=2.5$ which is the fractal dimensionality of percolation clusters on 3D lattices. The highest peak for $p<p_c$ appeared near the value $D=2.0$ which is the dimensionality of 3D clusters for $p<p_c$. This shows that the linking mechanism characterizes the fractal-growth process of networks not only on 2D lattices but also on 3D lattices.

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